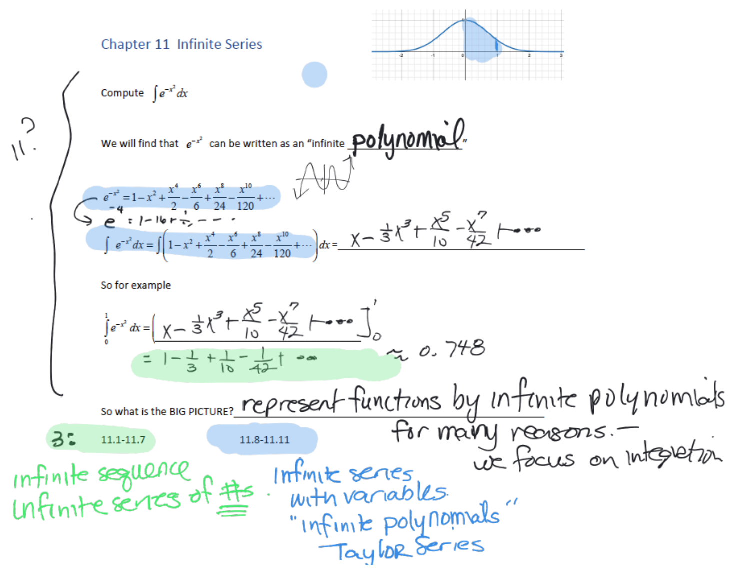
Math 5B 11.8-11.11 Notes: Power Series

Recall our original goal:


Chapter 11 Infinite Series

Compute $\int e^{-x^2}\,dx$

We will find that e^{-x^2} can be written as an "infinite polynomial"

$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \cdots$

$\int e^{-x^2}\,dx = \int\left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \cdots\right)dx = x - \frac{1}{3}x^3 + \frac{x^5}{10} - \frac{x^7}{42} + \cdots$

So for example

$\int_0^1 e^{-x^2}\,dx = \left[x - \frac{1}{3}x^3 + \frac{x^5}{10} - \frac{x^7}{42} + \cdots\right]_0^1 = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \cdots \approx 0.748$

So what is the BIG PICTURE? represent functions by infinite polynomials for many reasons. — we focus on integration

3: 11.1-11.7 — infinite sequence, infinite series of #s

11.8-11.11 — infinite series with variables. "infinite polynomials" Taylor Series

Also recall your HW from 11.6:

45. (a) Show that $\sum_{n=0}^{\infty} x^n/n!$ converges for all x.

(b) Deduce that $\lim_{n\to\infty} x^n/n! = 0$ for all x.

11.8 Power Series

A power series in x : $\sum_{n=0}^{\infty} c_n x^n$

We are often asked to determine for what values of x the given series converges, the Radius of Convergence and Interval of Convergence

Examples:

$$\sum_{n=0}^{\infty} x^n$$

And what is the sum?

$$\sum_{n=0}^{\infty} 3^n x^n$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sum_{n=0}^{\infty} n! x^n$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n (n+1)}$$

For power seres

of the form $\sum_{n=0}^{\infty} c_n x^n$, one of 3 things can happen:

1)____

2____

3____

Extending the above idea, we will consider power series "centered at a" : $\sum_{n=0}^{\infty} c_n (x-a)^n$

4 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

Examples: Find the interval and radius of convergence

$$\sum_{n=0}^{\infty}(x-3)^n$$

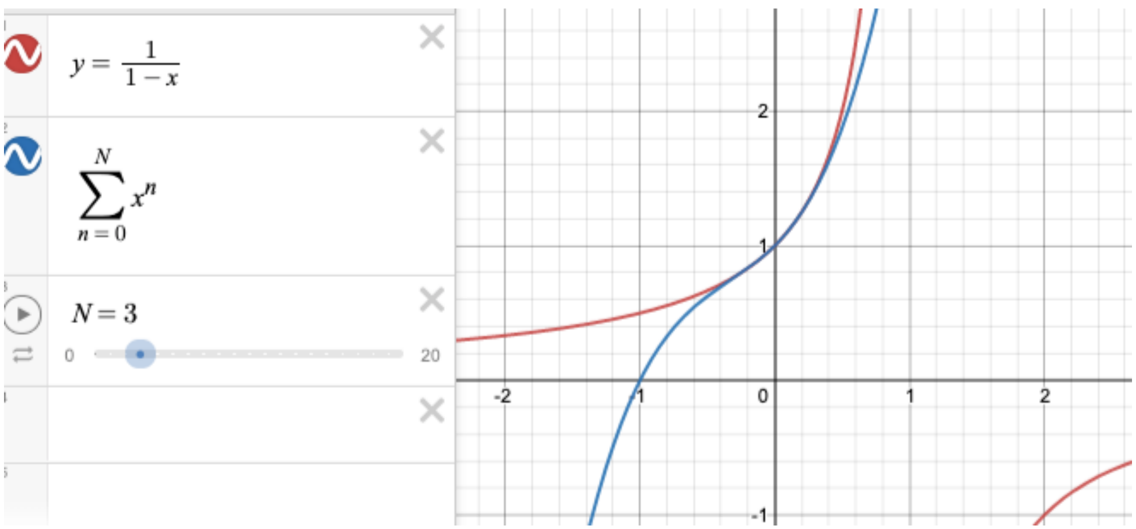
$$\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}(x-2)^n$$

11.10 and 11.9 Part 1: Generating Taylor and Maclaurin Series using the definition

In the last section we found that $\sum_{n=0}^{\infty} x^n$ converges for $|x|<1$. Not only that but we are able to find the exact sum in this case.

That is, if $|x|<1$, $f(x)=1+x+x^2+x^3+......=\sum_{n=0}^{\infty} x^n$ = __________

What does this mean? Graph $y=\dfrac{1}{1-x}$ and $y=1+x+x^2+x^3+......=\sum_{n=0}^{\infty} x^n$ on the same axes using Desmos. (To graph the second equation on Desmos, use the Desmos keyboard, functions, misc to get the summation symbol, then put in $y=\sum_{n=0}^{N} x^n$. Click on "add slider for N". You can change the range for n to start at zero by clicking on the endpoints of the shown slider.)



$S_4 = 1+x+x^2+x^3$

Can we do this for any function? Given a function f(x): Does f have a power series representation? (11.10 ii) If so, how do we find it? (11.10i)

We start by answering the second question first. Suppose f **does** have a power series representation. Let

$$P(x)=\sum_{n=0}^{\infty}c_n(x-a)^n=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+c_4(x-a)^4+c_5(x-a)^5+\dots$$

Our goal is to find the values for c_i such that $P(x)=$ ____________________ for all $|x|<R$.

If $P(x)=f(x)$, then it would follow that: $P'(x)=f'(x)$, $P''(x)=f''(x)$, $P'''(x)=f'''(x)$,...etc. In this case, $P(a)=f(a)$, $P'(a)=f'(a)$, $P''(a)=f''(a)$, $P'''(a)=f'''(a)$,... etc. Using these conditions, we can find the values of c_i

Taking derivatives of $P(x)$, which we will use for applying the conditions $P^{(n)}(a)=f^{(n)}(a)$ and thus finding. c_n

$P(x)=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+c_4(x-a)^4+c_5(x-a)^5+\dots$	$P(a)=$ ________	$c_0=$ ________
$P'(x)=c_1+2c_2(x-a)+3c_3(x-a)^2+4c_4(x-a)^3+5c_5(x-a)^4+\dots$	$P'(a)=$ ________	$c_1=$ ________
$P''(x)=2c_2+3\bullet 2c_3(x-a)+4\bullet 3c_4(x-a)^2+5\bullet 4c_5(x-a)^3+\dots$	$P''(a)=$ ________	$c_2=$ ________
$P'''(x)=$ ____	$P'''(a)=$ ________	$c_3=$ ________
$P^{(4)}(x)=$ ____	$P^{(4)}(a)=$ ________	$c_4=$ ________
	$\vdots$	$\vdots$
Following the pattern, we find	$P^{(n)}(a)=$ ________	$c_n=$ ________

In general we find: $c_n = \dfrac{f^{(n)}(a)}{n!}$ which we substitute into the power series

$P(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5 + \ldots$ to get the following:

IF $f(x)$ has a power series representation for $|x| < R$, it must be in the form:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This is called the *Taylor Series of* $f(x)$ *about x=a* (or centered at x=a)

In the specific case where a=0, the Taylor Series is given the special name *Maclaurin Series*.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

To find the Taylor Series representation of $f(x)$ we need to compute $f(a),\ f'(a),\ f''(a),\ f'''(a),\ f^{(4)}(a), \ldots$ and if possible find pattern for $f^{(n)}(a)$ and to substitute those values into the Taylor Series formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

11.10 and 11.9 Part 2: Generating Power Series from the Definition

To find the Taylor Series representation of $f(x)$ we need to compute $f(a),\quad f'(a),\quad f''(a),\quad f'''(a),\quad f^{(4)}(a),\ldots$ and if possible find pattern for $f^{(n)}(a)$ and to substitute those values into the Taylor Series formula:

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f^{(4)}(a)}{4!}(x-a)^4+\cdots=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\cdots=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

Example: Find the Maclaurin Series for $f(x)=\dfrac{1}{1-x}$

First: Find a patter for __________

	at x	at x=a=0
f		
f '		
f ''		
f '''		
f^n	$f^n(x)=$	$f^n(0)=$

Example: Find the Maclaurin Series of $f(x) = e^x$

First, try to find a pattern for $f^{(n)}(a) = f^{(n)}(0)$ since Marlaurin series implies a=0.

	at x	at x=a=0
f		
f '		
f ''		
f '''		
f^n	$f^n(x)=$	$f^n(0)=$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(n)}(a)}{4!}(x-a)^4 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

becomes

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

which leads to

$$e^x =$$

Find the interval of convergence: $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{n}\right| = \lim_{n\to\infty}$

Look at the graph of $y = e^x$ and $y = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ on the same axes using Desmos

Example: Find the Taylor Series for $f(x) = \ln x$ about a=2

First, try to find a pattern for $f^{(n)}(a) = f^{(n)}(2)$

	at x	at x=a=2
f		
f '		
f ''		
f '''		
f^n	$f^n(x)=$	$f^n(2)=$

Taylor Series Formula:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

For a=2:

$$f(x) = f(2) \ + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \frac{f^{(4)}(2)}{4!}(x-2)^4 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

leads to

$$\ln(x) = \quad + \quad (x-2) + \frac{\quad}{2!}(x-2)^2 + \frac{\quad}{3!}(x-2)^3 + \frac{\quad}{4!}(x-2)^4 + \ldots$$

$$\ln(x) = \ln(2) \ + \sum_{n=1}^{\infty} \underline{\qquad\qquad}$$

The interval of convergence can be shown to be (0, 4] (check)

Example: Find the Maclaurin Series of $f(x) = \cos(x)$

First, try to find a pattern for $f^{(n)}(a)$

	at x	at x=a=0
f		
f `		
f `′		
f `′′		
f ⁿ	f ⁿ(x)=	f ⁿ(0)=

Note: If a pattern cannot be found, we have to just write out terms as needed.
Then

$$\cos(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Leads to

$\cos(x) =$

Which shows a pattern after all:

$\cos(x) =$

Interval of Convergence: $(-\infty, \infty)$

Example: Find the Maclaurin Series for $f(x) = \dfrac{1}{(1+x)^4}$

(Note: Binomial Series can be done directly, without formula)

11.10 and 11.9 Part 3: Generating New Series from Known Series

Given a convergent power series, there are many operations we can perform which will create a new convergent power series. Because these are not finite sums, the fact that these operations are permissible, maintaining the equality, would need to be proved.

Substitution (may change the Radius of Convergence)

Example: Substitute –x for x in the series $\frac{1}{1-x}=1+x+x^2+x^3+......=\sum_{n=0}^{\infty}x^n$ to create a new series.

$$\frac{1}{1-x}=1+x+x^2+x^3+......=\sum_{n=0}^{\infty}x^n \qquad for \quad |x|<1$$

$$\frac{1}{1-(\quad)}=1+(\quad)+(\quad)^2+(\quad)^3+......=\sum_{n=0}^{\infty}(\quad)^n \qquad for \quad |(\quad)|<1$$

$$=\sum_{n=0}^{\infty} \qquad\qquad for \quad |x|<1$$

Example: Substitute 3x for x in the series $\frac{1}{1-x}=1+x+x^2+x^3+......=\sum_{n=0}^{\infty}x^n$ to create a new series.

$$\frac{1}{1-x}=1+x+x^2+x^3+......=\sum_{n=0}^{\infty}x^n \qquad for \quad |x|<1$$

$$\frac{1}{1-(\quad)}=1+(\quad)+(\quad)^2+(\quad)^3+......=\sum_{n=0}^{\infty}(\quad)^n \qquad for \quad |(\quad)|<1$$

$$=\sum_{n=0}^{\infty} \qquad\qquad for \quad |x|<$$

Example: Substitute x^2 for x in the series for cos(x) to create a new series.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad for \quad (-\infty, \infty)$$

$$\cos(\quad) = 1 - \frac{(\quad)^2}{2!} + \frac{(\quad)^4}{4!} + \ldots\ldots = \sum_{n=0}^{\infty} (-1)^n \frac{(\quad)^{2n}}{(2n)!} \quad = \sum_{n=0}^{\infty}$$

Multiplication by ax^n - (does not change the Radius of Convergence)

Example: Multiply the series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots = \sum_{n=0}^{\infty} x^n$ by 2x to create a new series.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots = \sum_{n=0}^{\infty} x^n \qquad for \quad |x| < 1$$

Differentiation (may change the Interval of Convergence, but not the radius)

Example: Differentiate the series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots = \sum_{n=0}^{\infty} x^n$ to create a new series.

$$\frac{1}{1-x} = \quad 1 + x + x^2 + x^3 + \ldots\ldots = \sum_{n=0}^{\infty} x^n \qquad for \quad |x| < 1$$

Notice what can happen to the index when differentiating. Note also that often times the index will be shifted so that the answer is in terms of x^n

Example: Differentiate the series for cos(x) to create a new series.

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\ldots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

Integration (may change the Interval of Convergence, but not the radius)

Example: Integrate the series for cos(x) to create a new series.

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\ldots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

Find C:

Multiplying/Dividing Series	Interval of Convergence is intersection, with denominator not equal zero.

Example: Find the Maclaurin series for $f(x) = e^x \sin x$

a) Direct Approach, using the definition and generating $f^{(n)}(0)$ if possible:

	at x	at x=a=0
f		
f '		
f ''		
f '''		
f^n	$f^n(x)=$	$f^n(0)=$

b) Using known series:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \tfrac{x^4}{4!} \cdots$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$e^x \sin x =$

Notice in this case, we do not have the general term so cannot write in sigma notation.
See book for example using division to get the Maclaurin series for $f(x) = \tan x$

Now that we have discussed the various operations we can use on known series to create new series, let us look at how we will actually make use of those operations. Typically we are not given a known series and told what operations to perform on it but instead we work in reverse. We are given a function for which we would like to create a power series representation. We can build the series directly, using the Taylor Series definition,

$$f(x)=f(a)\ +f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f^{(4)}(a)}{4!}(x-a)^4+\ldots=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

but if we can see how the given function could be created from a function with a known, or easily found series representation, that might be quicker.

Example: Find the Maclaurin series for $f(x)=e^{-x^2}$

$$e^x=\sum_{n=0}^{\infty}\frac{1}{n!}x^n=1+x+\tfrac{x^2}{2!}+\tfrac{x^3}{3!}+\tfrac{x^4}{4!}\cdots$$

$$e^{-x^2}=\sum_{n=0}^{\infty}\frac{1}{n!}(\quad)^n=1+(\quad)+\tfrac{(\ )^2}{2!}+\tfrac{(\ )^3}{3!}+\tfrac{(\ )^4}{4!}\cdots$$

Example: Find a power series representation for $f(x) = \frac{2}{3x+4}$

Note: On the problems in 11.9, all the problems build using the geometric series.

Example: Find a power series representation for $f(x)=\tan^{-1}\left(x^2\right)$

Example : Find the Maclaurin Series for $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

(Note: pattern sometimes begins for n>0)

11.10 and 11.9 Part 4: Using Taylor Series for Estimation and Integration.

FINALLY, in this part we will do what we set out to do in this chapter! We will see how Taylor series can be used for estimation and integration

Application Example: Use the Maclaurin series for $f(x) = e^x$

to **estimate** a) $e^{-0.4}$ b) e^{-3} with error < 0.01.

Recall: $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$ for all x.

---

a) $e^{-0.4} = 1 + (\quad) + \frac{1}{2!}(\quad)^2 + \frac{1}{3!}(\quad)^3 + \frac{1}{4!}(\quad)^4 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}(\quad)^n \quad = 1 - 0.4 + 0.08 - 0.0107 + 0.00107 + \ldots$

^

Recall AST error estimate, ...

since the 5th term is the first term < 0.01, our estimate is taken from the ______th partial sum.

$$e^{-0.4} \approx S_4 = 1 - 0.4 + 0.08 - 0.0107 \approx 0.669$$ (calculator value 0.670320046)

$$f(x) = \left(\frac{(-0.4)^{(x)}}{(x)!} \right)$$

x_1	$f(x_1)$	$\sum_{j=0}^{x_1} f(j)$
0	1	1
1	-0.4	0.6
2	0.08	0.68
3	-0.010666667	0.66933333
4	0.0010666667	0.6704
5	-8.533333×10^{-5}	0.67031467
6	5.688889×10^{-6}	0.67032036

b) $e^{-3} = 1 + (\quad) + \frac{1}{2!}(\quad)^2 + \frac{1}{3!}(\quad)^3 + \frac{1}{4!}(\quad)^4 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}(\quad)^n$

$= 1 - 3 + 4.5 - 4.5 + 3.375 - 2.025 + 1.0125 - 0.4339 + 0.1626 - 0.0044 + \ldots$

^

$$e^{-3} \approx S_9 \approx 0.0533$$ (calculator value 0.049787)

Question: Why does part b require many more terms to provide an estimate with the same error?

Taylor Series Converge faster for values of x near the base point a.

It is desirable to choose Taylor Series with a base point close to region of interest. See handout on 5B page about choosing base point.

Application Example: Compute $\int_0^{1/2} e^{-x^2}\,dx$ with $|R_n| < 0.01$

In a previous example, we found:

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

So

$$\int_0^{1/2} e^{-x^2}\,dx = \int_0^{1/2} \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \cdots\right) dx = \int_0^{1/2} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}\,dx$$

(Wolframalpha: $\int_0^{1/2} e^{-x^2}\,dx \approx 0.4612810064$)

<u>Application Example</u>: Compute $\int_0^{1/2} \tan^{-1}\left(x^2\right)dx$ with error < 0.0001

Previously, we found the series for $f(x) = \tan^{-1}\left(x^2\right)$

$$\tan^{-1}\left(x^2\right) = x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

$$\int_0^{1/2} \tan^{-1}\left(x^2\right)dx = \frac{1}{1\bullet 3\bullet 2^3} - \frac{1}{3\bullet 7\bullet 2^7} + \frac{1}{5\bullet 11\bullet 2^{11}} - \frac{1}{7\bullet 15\bullet 2^{15}} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(4n+3)2^{4n+3}}$$

So $\int_0^{1/2} \tan^{-1}\left(x^2\right)dx = 0.0416667\text{-}0.000372\text{+}0.00000888\text{-}\ldots$

$$\int_0^{1/2} \tan^{-1}\left(x^2\right)dx \approx 0.0416667\text{-}0.000372\text{=}0.0412947$$

One last thing:

Recall: We began with two questions:

Does f have a power series representation? (11.10 ii)

If so, how do we find it? (11.10i)

We have been working on the answer to the second question. We will now look at the first question and discuss the error involved in using the Taylor series to approximate f(x).

11.10 and 11.9 Part 5: Proving the "=".

Does f actually have a power series representation, that is, does the Taylor Series corresponding to $f(x)$ actually converge TO $f(x)$?

Back to the definition of convergence of series. Construct the sequence of partial sums.

$S_1 = f(a)$

$S_2 = f(a) + f'(a)(x-a)$

$S_3 =$ ________________________________

$\vdots$

$$S_n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

$$S_{n+1} = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots\cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \quad = \quad T_n(x)$$

Recall that for the series to converge, $\lim_{n\to\infty} S_n = S$, where S is the (finite) sum.

Thus, we need to show that: $\lim_{n\to\infty} S_n =$ ________ or $\lim_{n\to\infty} S_{n+1} = \lim_{n\to\infty} T_n(x) =$ _________

Since we want

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots\cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \quad + \quad \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \frac{f^{(n+2)}(a)}{(n+2)!}(x-a)^{n+2} + \cdots$$

$$= T_n(x) + \quad \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \frac{f^{(n+2)}(a)}{(n+2)!}(x-a)^{n+2} + \cdots$$

to show $\lim_{n\to\infty} S_{n+1} = \lim_{n\to\infty} T_n(x) = f(x)$ we need to show that

$$\lim_{n\to\infty}\left(\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}+\frac{f^{(n+2)}(a)}{(n+2)!}(x-a)^{n+2}+\cdots\right)= \_\_\_\_\_\_\_\_\_\_\_\_$$

Consider the following extension of the MVT*... It can be shown, in Advanced Calculus, that for some c between x and a,

$\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}+\frac{f^{(n+2)}(a)}{(n+2)!}(x-a)^{n+2}+\cdots\cdots=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between x and a.

Thus, we need only show

$$\lim_{n\to\infty}\left(\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right)=0$$

for any c between x and a.

We know $\lim_{n\to\infty}\frac{1}{(n+1)!}(x-a)^{n+1}= \_\_\_\_\_\_\_\_\_\_\_\_$,(why?) so we only need to show that $f^{(n+1)}(c)$ is bounded for c between x and a, that is $\left|f^{(n+1)}(c)\right|\le M$,

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Example: Show that the Maclaurin Series for $f(x)=\sin x$ converges to $\sin x$.

11.11 Error for Taylor Polynomial Approximation

So far, the only error estimates we have discussed for series are for the ______________________

and the ________________________. What if neither or these apply to the Taylor Series for a given function, What can be said about the error?

For a function given by its Taylor Series:

$$f(x) = T_n(x) + \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \frac{f^{(n+2)}(a)}{(n+2)!}(x-a)^{n+2} + \cdots$$

let $R_n(x)$ represent the error associated with using the nth degree Taylor Polynomial to approximate $f(x)$. Then

$$R_n(x) =$$

But we found in the previous section that for some c between x and a

$$\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \frac{f^{(n+2)}(a)}{(n+2)!}(x-a)^{n+2} + \cdots\cdots = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

which means

$$R_n(x) =$$

If $f^{(n+1)}(c)$ is bounded for c between x and a, that is if $\left|f^{(n+1)}(c)\right| \le M$, then

$$\left|R_n(x)\right| \le$$

Example:

(a) Approximate the function $f(x) = x^{2/3}$ by T₃(x), the third degree Taylor Polynomial centered at a=1.

(b) Use $T_3(x)$ to approximate $0.8^{2/3}$

(c) Use Taylor's inequality to estimate the accuracy of the approximation when x lies in the interval $0.7 \le x \le 1.3$

(a)

$T_3(x) = f(a) + \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

$= f(1) + \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

(b) $0.8^{2/3} \approx$

(c) $\left|R_n(x)\right| \le \frac{M}{(n+1)!}(x-a)^{n+1}$, where $\left|f^{(n+1)}(c)\right| \le M$ for some c between x and a, so

$\left|R_3(x)\right| \le$ ____________________ , where ______________ for some c between x and 1,

where $0.7 \le x \le 1.3$.

(Note: this is more powerful than the alternating series error estimate because it can be used to find what values of x are allowed with a given n and error tolerance)